

# HOMOTOPY K3 SURFACES CONTAINING $\Sigma(2, 3, 7)$

RONALD FINTUSHEL & RONALD J. STERN

## 1. Introduction

An interesting question in 4-dimensional topology is whether each irreducible simply connected smooth 4-manifold other than  $S^4$  must admit a complex structure. One technique which has been suggested for answering this question is to try to produce examples which have all of their Donaldson polynomials [3] vanishing, and then use Donaldson's theorem that complex algebraic surfaces have nontrivial polynomial invariants. A natural place to begin study is among smooth manifolds with the homotopy type of a K3 surface; we refer to such manifolds as homotopy K3 surfaces. Kodaira [7] has produced a family of homotopy K3 surfaces by performing logarithmic transforms on the fibers of elliptic K3's but these manifolds all have complex structures. (Their diffeomorphism types have been studied recently by Friedman and Morgan.) Also, there were many (unpublished) examples of homotopy K3 surfaces constructed about a decade ago by Kirby calculus pictures.

A common aspect of many of these latter examples is that they admit an embedding of the Brieskorn homology 3-sphere  $\Sigma(2, 3, 7)$ , which may be described as the link of a complex algebraic singularity  $\{(x, y, z) \in \mathbb{C}^3 : x^2 + y^3 + z^7 = 0\} \cap S^5$ , or, equivalently, as the result of  $-1$  surgery on the right-handed trefoil knot. (The Poincaré homology sphere,  $\Sigma(2, 3, 5)$ , is the result of  $+1$  surgery on the right-handed trefoil.) In this article we shall show

**Theorem 1.1.** *Any homotopy K3 surface which admits an embedding of  $\Sigma(2, 3, 7)$  has a nontrivial Donaldson polynomial invariant of degree 10.*

We will give a fairly elementary proof of this fact based on Donaldson's study of 4-manifolds whose intersection form has one or two positive parts [2] and on our study of the representation space of  $\Sigma(2, 3, 7)$  [4]. In particular, our calculations of Donaldson's invariant use no algebraic geometry.

---

Received October 7, 1989 and, in revised form, June 6, 1990. The authors were partially supported by National Science Foundation grants DMS8802412 and, DMS870413, respectively.

The philosophy of our proof is closely related to forthcoming work of Donaldson which gives his relative polynomial invariants in terms of Floer's instanton homology theory. However, our result neither uses the generality of that theory, nor does it admit any obvious generalizations in that direction. The proof uses an analysis of the effect on anti-self-dual moduli spaces of letting a metric on the homotopy K3 surface degenerate along our homology sphere. It then applies rather specific knowledge about the representation space of  $\pi_1(\Sigma(2, 3, 7))$  into  $SU(2)$ .

Our result is also a key step in Akbulut's construction of a fake relative smooth structure on a compact contractible 4-manifold [1]. Akbulut's construction produces a homotopy K3 surface  $M$  containing  $\Sigma(2, 3, 7)$  and contractible 4-manifold  $W$  together with a self-diffeomorphism  $f$  of  $\partial W$  which extends to a self-homeomorphism of  $W$  such that either

- (1)  $M$  has all its Donaldson polynomial invariants trivial, or
- (2) there is no self-diffeomorphism of  $W$  extending  $f$ .

Our theorem then implies that (1) is false; so Akbulut's construction gives (2).

## 2. Donaldson's polynomial invariant

Let us begin by quickly reviewing the construction of Donaldson's polynomial invariant [3]. Suppose we have a simply connected closed 4-manifold  $M^4$ , where  $b_2^+(M)$  is odd and  $> 1$ . The moduli space  $\mathcal{M}_k(M)$  of anti-self-dual connections on the  $SU(2)$  bundle over  $M$  with  $c_2 = k$  has formal dimension  $8k - 3(1 + b_2^+) = 2d$ . For a generic choice of Riemannian metric on  $M$ , the moduli space,  $\mathcal{M}_k(M)$ , if nonempty, will be a manifold of this dimension [6], [2]. (Here, as usual, we must use Sobolev spaces of connections; see [2] for details.) If  $k > \frac{3}{4}(1 + b_2^+)$ , Donaldson [3] defines a polynomial  $q_k(M)$  in the polynomial algebra  $P[H_2(M; \mathbf{Z})]$  as follows. Let  $\mathcal{B}(M)$  denote the space of gauge equivalence classes of connections on  $SU(2)$  bundles over  $M$ . For  $z \in H_2(M; \mathbf{Z})$  choose an embedded oriented surface  $S$  representing  $z$  with an open neighborhood  $N$  such that  $H_2(N; \mathbf{Z}) = \mathbf{Z}[S]$ . If  $r_S: \mathcal{B}(M) \rightarrow \mathcal{B}(N)$  denotes the restriction map and  $\mathcal{B}_N^\dagger$  denotes the space of gauge equivalence classes of  $SU(2)$  connections over  $N$  which are either irreducible or trivial, then Donaldson shows in an appendix to [3] that the surface  $S$  representing  $z$  can be chosen so that  $\coprod_{l=0}^k \mathcal{M}_l(M)$  is contained in  $r_S^{-1}(\mathcal{B}_N^\dagger)$ . Furthermore, there is a complex line bundle  $L_S$  over  $\mathcal{B}_N^\dagger$  with a section so that when pulled back over  $r_S^{-1}(\mathcal{B}_N^\dagger)$  it gives a section  $\sigma_S$  of  $r_S^*(L_S)$  whose zero set  $V_S$  is

a codimension-2 submanifold of  $r_S^{-1}(B_N^\dagger)$  which meets all of the moduli spaces  $\mathcal{M}_l$  ( $l \leq k$ ) transversely. In particular, since  $\mathcal{M}_0(M) = \{\Theta\}$ , where  $\Theta$  is the trivial  $SU(2)$  connection, transversality means that  $\Theta \notin V_S$ . (The map  $z \mapsto c_1(r_S^*(L_S))$  is the map  $\mu: H_2(M; \mathbf{Z}) \rightarrow H^2(\mathcal{B}(M); \mathbf{Z})$  constructed in [2].)

Given homology classes  $z_1, \dots, z_d \in H_2(M; \mathbf{Z})$ , represent them by surfaces  $S_1, \dots, S_d$  in  $M$  chosen as above. It is possible to pick the surfaces to be in general position and have their open neighborhoods  $N_i$  so that all triple intersections  $N_i \cap N_j \cap N_k$  are empty. The intersection  $V_{S_1} \cap \dots \cap V_{S_d} \cap \mathcal{M}_k(M)$  will then be discrete, and the condition  $k > \frac{3}{4}(1+b_2^+)$  will imply that it is compact. (The  $V_{S_i}$  are also chosen to have transverse multiple intersections.) The Donaldson polynomial invariant is defined to be

$$q_k(M)(z_1, \dots, z_d) = \#(V_{S_1} \cap \dots \cap V_{S_d} \cap \mathcal{M}_k(M)),$$

where  $\#$  denotes a count with signs. Donaldson proves that  $q_k(M)$  depends only on the smooth structure of  $M$ . (See [3].)

Now let  $M$  denote a smooth oriented homotopy K3 surface. This means that  $\pi_1(M) = 0$  and that the intersection form of  $M$  is  $2E_8 \oplus 3H$  of rank 22 and signature  $-16$ .

**Lemma 2.1.**  $\Sigma(2, 3, 7)$  embeds in a K3 surface, splitting it into submanifolds with intersection forms  $E_8 \oplus H$  and  $E_8 \oplus 2H$ .

*Proof.* This follows from [8] where it is shown that the triangle singularity  $D_{2,3,7} = \Sigma(2, 3, 7)$  embeds in a K3 surface with smoothing given by the  $E_{10}$  diagram.

**Lemma 2.2.** If  $\Sigma(2, 3, 7)$  embeds in a homotopy K3 surface  $M$ , it splits  $M$  into smooth oriented submanifolds  $M = X \cup Y$ , where the intersection form of  $X$  is  $E_8 \oplus H$ , and the intersection form of  $Y$  is  $E_8 \oplus 2H$ .

*Proof.* The  $\mu$ -invariant of  $\Sigma(2, 3, 7)$  is nonzero; so the intersection forms of  $X$  and  $Y$  must each have an  $E_8$  summand. Now by Lemma 2.1,  $\pm\Sigma(2, 3, 7)$  bounds manifolds which have intersection forms  $E_8 \oplus H$  and  $E_8 \oplus 2H$ . Thus, if either  $X$  or  $Y$  had the intersection form  $E_8$ , we could construct a simply connected 4-manifold with intersection form  $2E_8 \oplus kH$  for  $k = 1$  or  $2$ , in contradiction to Donaldson's Theorems B and C of [2]. q.e.d.

Next we present information about the representations of the fundamental group of  $\Sigma = \Sigma(2, 3, 7)$ . We shall use the same notation for (the conjugacy class of) a representation  $\pi(\Sigma) \rightarrow SU(2)$  as for the gauge equivalence class of flat connections over  $\Sigma$  which it induces. Consider an  $SU(2)$  connection  $A$  over  $\pm\Sigma \times \mathbf{R}$  with finite action. It must be asymptotically

flat with limiting connections  $\rho, \sigma$  as  $t \rightarrow \pm\infty$ . Let  $\mathcal{M}_{\pm\Sigma}(\rho, \sigma)$  denote the moduli space of anti-self-dual  $SU(2)$  connections over  $\pm\Sigma \times \mathbf{R}$  with these asymptotic conditions. (It is imperative in this situation to use connections with exponential decay and a corresponding group of gauge transformations. See [5].) Let  $\theta$  denote the trivial representation of  $\pi_1(\Sigma)$ . The next proposition follows directly from computations in [4].

**Proposition 2.3.** *Let  $\Sigma = \Sigma(2, 3, 7)$ . Then up to conjugacy there are two nontrivial representations  $\alpha, \beta: \pi_1(\Sigma) \rightarrow SU(2)$ , and the (mod 8) dimensions of the corresponding moduli spaces of anti-self-dual connections on  $\pm\Sigma \times \mathbf{R}$  are:*

$$\begin{aligned} \dim \mathcal{M}_{\Sigma}(\alpha, \theta) &\equiv 2, & \dim \mathcal{M}_{-\Sigma}(\alpha, \theta) &\equiv 3, \\ \dim \mathcal{M}_{\Sigma}(\beta, \theta) &\equiv 6, & \dim \mathcal{M}_{-\Sigma}(\beta, \theta) &\equiv 7. \end{aligned}$$

Now let  $M = X \cup Y$  as in Lemma 2.2. Choose homology classes  $z_1, z_2, z_3, z_4 \in H_2(X; \mathbf{Z}) = E_8 \oplus H$  such that the pair  $z_1, z_2 \in E_8$  satisfies  $z_1^2 = z_2^2 = 2$  and  $z_1 \cdot z_2 = 1$ , and the pair  $z_3, z_4 \in H$  satisfies  $z_3^2 = z_4^2 = 0$  and  $z_3 \cdot z_4 = 1$ . Similarly, choose  $z_5, \dots, z_{10} \in H_2(Y; \mathbf{Z}) = E_8 \oplus 2H$  so that  $z_5, z_6$  form a pair in  $E_8$  and  $z_7, z_8$  and  $z_9, z_{10}$  form pairs in the two copies of  $H$ . For each  $z_i$  choose an oriented surface  $S_i$  in  $X$  or  $Y$  as in the definition of Donaldson's polynomial. We may suppose that the  $S_i$  are in general position. Note that  $\dim \mathcal{M}_4(M) = 20$ .

Consider now the result of collapsing the homology sphere  $\Sigma$  to a point. After removing the singular point, we are left with a disjoint union  $X_+ \amalg Y_-$ , where  $X_+ = X \cup (\partial X \times [0, \infty))$  and  $Y_- = (\partial Y \times (-\infty, 0]) \cup Y$ . According to [11, §8], for generic metrics  $g_X$  and  $g_Y$  any moduli spaces of anti-self-dual connections  $\mathcal{M}_{X_+}(\rho)$  and  $\mathcal{M}_{Y_-}(\rho)$  are manifolds. Since we are dealing with finite action solutions of the anti-self-duality equations, these solutions must be asymptotically flat. Here " $\rho$ " denotes the representation corresponding to the flat asymptotic connection.

Fix a generic metric  $g$  on  $M$ . We can then choose codimension-2 submanifolds  $V_{S_1} \cdots V_{S_{10}}$  whose multiple intersections with the moduli spaces  $\mathcal{M}_l(M, g)$ ,  $l \leq 4$ , are transverse. Furthermore, the  $V_{S_i}$  can also be chosen to have transverse multiple intersections with any components of  $\mathcal{M}_{X_+}(g_X, \rho) \amalg \mathcal{M}_{Y_-}(g_Y, \rho)$  of total formal dimension  $\leq 20$ . (We can view the  $V_{S_i}$  as living in  $\mathcal{B}^*(X_+ \amalg Y_-)$  because the restriction maps  $\mathcal{B}(M) \rightarrow \mathcal{B}(N_i)$  factor through  $\mathcal{B}(M \setminus \Sigma)$ .)

Now consider a sequence  $\{g_n\}$  of generic metrics on  $M$  with  $g_0 = g$  which converge to the (necessarily singular) metric  $g_X \vee g_Y$  on  $M/\sim$ , where  $\sim$  collapses  $\Sigma$  to a point. Since the cone on  $\Sigma$  minus the cone

point is conformally equivalent to a cylinder  $\Sigma \times \mathbf{R}$ , we get our limiting metrics  $g_X$  and  $g_Y$  on  $X_+$  and  $Y_-$ . The  $V_{S_i}$ , being submanifolds of the Hilbert manifold  $\mathcal{B}^*(M)$ , have tubular neighborhoods. For each  $i$ , let  $\mathcal{V}_{S_i}^{(n)}$  be such a sequence of tubular neighborhoods with radii  $\rightarrow 0$  as  $n \rightarrow \infty$ . Then, using [3], for each  $1 \leq n \leq \infty$  we can perturb the sections  $\sigma_i$  to obtain new codimension-2 zero sets,  $V_{S_i}^{(n)}$ , contained in  $\mathcal{V}_{S_i}^{(n)}$ , so that the intersection of any  $p$  of the  $V_{S_i}^{(n)}$  have empty intersection with any moduli space  $\mathcal{M}_l(M, g_n)$  of formal dimension less than  $2p$ . Donaldson also shows that the polynomial invariant can be computed from these  $V_{S_i}^{(n)}$  after further, arbitrarily small, perturbation making all the intersections transverse.

Suppose now that  $q_4(M)(z_1, \dots, z_{10}) \equiv 1 \pmod{2}$ . In particular, for each  $n$ ,  $V_{S_1}^{(n)} \cap \dots \cap V_{S_{10}}^{(n)} \cap \mathcal{M}_4(M, g_n)$  is nonempty. Choose an anti-self-dual connection  $A_n \in V_{S_1}^{(n)} \cap \dots \cap V_{S_{10}}^{(n)} \cap \mathcal{M}_4(M, g_n)$  for each  $n$ . By Uhlenbeck's Compactness Theorem [12], there are limiting anti-self-dual connections  $A_X$  over  $X_+ \setminus \{x_1, \dots, x_r\}$ ,  $A_Y$  over  $Y_- \setminus \{y_1, \dots, y_s\}$ , and limiting instantons at the points  $x_i$  and  $y_j$ . Both  $A_X$  and  $A_Y$  are asymptotically flat with limiting flat connections  $\rho_X$  and  $\rho_Y$ . In addition there is the possibility that, in the limit, curvature is lost at the neck between  $X_+$  and  $Y_-$ . In that case one also has flat connections  $\rho_X = \rho_0, \rho_1, \dots, \rho_m = \rho_Y$  and limiting nontrivial anti-self-dual connections  $B_i \in \mathcal{M}_{\pm\Sigma}(\rho_{i-1}, \rho_i)$  for  $i = 1, \dots, m$ . Let  $n_X$  and  $n_Y$  denote the formal dimensions (i.e., the dimensions given by the index theorem) of the components of the moduli spaces containing  $A_X$  and  $A_Y$ , and for each  $i = 1, \dots, m$  let  $n_i > 0$  be the formal dimension of the component of  $\mathcal{M}_{\pm\Sigma}(\rho_{i-1}, \rho_i)$  containing  $B_i$ . Also, let  $T \geq 0$  be the number of  $\rho_i = \theta$ . Then, counting dimensions, we have:

$$20 = n_X + n_Y + 8(r + s) + \sum n_i + 3T.$$

Since the surfaces  $S_i$  are in general position in  $M$ , no point lies on more than two surfaces. So on  $X_+$  the points  $x_1, \dots, x_r$  lie on at most  $2r$  of the  $S_i$ . Recall that a connection lives in  $V_{S_i}$  if and only if when restricted to the open neighborhood  $N_i$  of  $S_i$  it lies in the zero set of the section  $\sigma_i$ . Each  $A_n \in V_{S_i}^{(n)}$ , which converge as sets to  $V_{S_i}$ ; so if no  $x_j \in S_i$ , then  $A_X \in V_{S_i}$ . Similarly, for  $k = 5, \dots, 10$ ,  $A_Y \in V_{S_k}$  if no  $y_l \in S_k$ . Suppose first that  $A_X$  is the trivial connection,  $\Theta_X$ , on  $X_+$ ; so  $n_X = -6$  and  $T \geq 1$ . But  $\Theta_X$  does not lie in any  $V_{S_i}$ ; so for  $i = 1, \dots, 4$ , and  $j = 1, \dots, r$  each  $S_i$  contains some point  $x_j$ . Thus

$r \geq 2$ . If  $A_Y$  is also trivial then  $n_Y = -9$ , and similarly  $s \geq 3$ . The above count of formal dimensions,

$$20 = -6 + (-9) + 8(r + s) + \sum n_i + 3T \geq 25 + \sum n_i + 3,$$

then gives a contradiction. If  $A_X$  is trivial but  $A_Y$  is nontrivial then  $n_Y \geq 0$  and  $A_Y$  lies in at least  $6 - 2s$  of the  $V_{S_k}$ 's. Since each  $V_{S_i}$  is codimension-2 this means that  $2(6 - 2s) \leq n_Y$ . The formal dimension count then gives the contradiction

$$20 \geq -6 + 2(6 - 2s) + 8(2 + s) + \sum n_i + 3 \geq 25 - 4s + \sum n_i.$$

A similar formal dimension count shows that we cannot have  $A_Y = \Theta_Y$  and  $A_X$  nontrivial so that neither  $A_X$  nor  $A_Y$  is trivial. Applying our formal dimension count once more we get

$$20 \geq 2(4 - 2r) + 2(6 - 2s) + 8(r + s) + \sum n_i + 3T.$$

Thus,  $r = s = 0$ ,  $T = 0$ , and  $\sum n_i = 0$ ; so  $m = 0$  and  $\rho_X = \rho_Y = \rho$ , say. Also  $n_X = 8$  and  $n_Y = 12$ . Arguing further we get:

**Proposition 2.4.** *If  $q_4(M)(z_1, \dots, z_{10}) \equiv 1 \pmod{2}$ , then there are connections  $A_X \in V_{S_1} \cap \dots \cap V_{S_4} \cap \mathcal{M}_{X_+}(\alpha)$  and  $A_Y \in V_{S_5} \cap \dots \cap V_{S_{10}} \cap \mathcal{M}_{Y_-}(\alpha)$ , where  $\dim \mathcal{M}_{X_+}(\alpha) = 8$  and  $\dim \mathcal{M}_{Y_-}(\alpha) = 12$ . Furthermore,  $\partial X = \Sigma$ , and for any sequence of connections  $\{A_n\}$  as above we have  $\rho = \alpha$ .*

*Proof.* If we get the asymptotic condition  $\rho$ , then  $\dim \mathcal{M}_{X_+}(\rho) = 8$ . Let  $B$  be any connection over  $\partial X \times \mathbf{R}$  (where  $\partial X = \pm \Sigma$ ), which tends asymptotically to  $\rho$  as  $t \rightarrow -\infty$  and to  $\theta$  as  $t \rightarrow +\infty$ . Grafting  $A_X$  to  $B$  as in [5] we obtain a connection  $A_X \# B$  over  $X_+$ , which is asymptotically trivial, and so the index of the anti-self-duality operator  $D_{A_X \# B}$  is  $8k - 3(1 + b_2^+(X)) \equiv -6 \pmod{8}$ . But also  $\text{Ind } D_{A_X \# B} = \text{Ind } D_{A_X} + \text{Ind } D_B = 8 + \text{Ind } D_B \equiv \text{Ind } D_B \pmod{8}$  since  $\beta$  is irreducible. It now follows from Proposition 2.3 that  $\partial X = +\Sigma$  and that  $\rho = \alpha$ . q.e.d.

Conversely, we have

**Proposition 2.5.** *Let  $\mathcal{M}_{X_+}(\alpha)$  and  $\mathcal{M}_{Y_-}(\alpha)$  be moduli spaces of anti-self-dual connections of dimensions equal to 8 and 12 respectively. Let  $z_i, S_i, i = 1, \dots, 10$ , be as above, and let  $m_\alpha = \#(V_{S_1} \cap \dots \cap V_{S_4} \cap \mathcal{M}_{X_+}(\alpha))$  and  $n_\alpha = \#(V_{S_5} \cap \dots \cap V_{S_{10}} \cap \mathcal{M}_{Y_-}(\alpha))$ . Then  $q_4(z_1, \dots, z_{10}) \equiv m_\alpha n_\alpha \pmod{2}$ .*

*Proof.* If  $A_X \in V_{S_1} \cap \dots \cap V_{S_4} \cap \mathcal{M}_{X_+}(\alpha)$  and  $A_Y \in V_{S_5} \cap \dots \cap V_{S_{10}} \cap \mathcal{M}_{Y_-}(\alpha)$ , then for metrics  $g$  on  $M$  close enough to  $g_X \vee g_Y$  there is a grafted anti-self-dual connection  $A_X \#_g A_Y$  in  $V_{S_1} \cap \dots \cap V_{S_{10}} \cap \mathcal{M}_k(M, g)$ , where

$8k - 3(1 + b_2^+(M)) = 8 + 12$ ; so  $k = 4$ . A study of this grafting process shows that for  $g$  close to  $g_X \vee g_Y$  each  $A \in \mathcal{M}_4(M, g)$  can be uniquely written as  $A = A_X \#_g A_Y$  (see [9]).

### 3. Proof of Theorem 1.1

Let  $M = X \cup Y$  be as in Lemma 2.2 and let  $S_1, \dots, S_{10}$  be the surfaces in  $X$  described in §2. The goal of this section is to build moduli spaces  $\mathcal{M}_{X_+}(\alpha)$  and  $\mathcal{M}_{Y_-}(\alpha)$  as described in Proposition 2.5 such that  $m_\alpha$  and  $n_\alpha$  are odd, so that  $q_4(M)(S_1, \dots, S_{10}) \equiv 1 \pmod{2}$ . The basic idea is to apply ideas of the proofs of Donaldson's Theorems B and C in [2]. In this case, instead of obtaining contradictions, we obtain information about the ends of the moduli space, which correspond to the asymptotics of  $X_+$  (or  $Y_-$ ). First we work with  $X_+$ . It follows from the work of Taubes [10], [11] that the moduli space  $\mathcal{M}_{X_+,2}(\theta)$  of  $c_2 = 2$  asymptotically trivial anti-self-dual connections over  $X_+$  is nonempty, and is therefore a 10-dimensional manifold when  $X_+$  is given a generic metric. We want to study the 2-manifold  $N^2 = V_{S_1} \cap \dots \cap V_{S_4} \cap \mathcal{M}_{X_+,2}(\theta)$ , where  $S_1, \dots, S_4$  are the surfaces described in §2. We need to examine the ends of  $N^2$ .

The ends of  $\mathcal{M}_{X_+,2}(\theta)$  correspond to the ways that sequences of anti-self-dual connections in  $\mathcal{M}_{X_+,2}(\theta)$  can converge to an anti-self-dual connection with a different  $c_2$  or asymptotic condition. For example, such a sequence could converge to

- (1) a  $c_2 = 1$  anti-self-dual connection  $A_\infty \in \mathcal{M}_{X_+,1}(\theta)$  together with an instanton at a point  $x \in X$ ,
- (2) the trivial connection  $\Theta_X$  together with a pair of instantons at points  $x$  and  $y$  in  $X$ , or
- (3) an anti-self-dual connection  $A_\rho \in \mathcal{M}_{X_+,1}(\rho)$ ,  $\rho$  a nontrivial asymptotic condition, together with an instanton over  $\partial X \times \mathbf{R}$  (where  $\partial X = \pm \Sigma$ ) which tends asymptotically to  $\rho$  as  $t \rightarrow -\infty$  and to  $\theta$  as  $t \rightarrow +\infty$ .

This description also indicates how the moduli space  $\mathcal{M}_{X_+,2}(\theta)$  is compactified. For details see [2].

If a sequence  $\{A_n\}$  in  $N^2$  converges to an  $A_\infty \in \mathcal{M}_{X_+,1}(\theta)$  together with an instanton at a point  $x \in X$ , then as in §2 the point  $x$  lies on at most two of the surfaces  $S_i$ , and so there are  $i_1 \neq i_2$  with  $A_\infty$  lying in the transverse intersection  $V_{S_{i_1}} \cap V_{S_{i_2}} \cap \mathcal{M}_{X_+,1}(\theta)$ . But  $V_{S_{i_1}} \cap V_{S_{i_2}}$  is codimension-4 and  $\dim \mathcal{M}_{X_+,1}(\theta) = 2$ ; so this situation cannot occur.

Suppose there is an end of  $N^2$  corresponding to a sequence  $\{A_n\}$  converging to  $\Theta_X$  together with instantons at  $x$  and  $y$  in  $X$ . If there is an  $S_i$  containing neither  $x$  nor  $y$ , then  $\Theta_X \in V_{S_i}$ . However as noted in §2,  $\Theta_X \notin V_{S_i}$  for any  $i$ . Thus the ends of  $N^2$  coming from sequences converging to  $\Theta_X$  corresponding to pairs  $\{x, y\} \in S^2(X)$  such that each  $S_i$  contains  $x$  or  $y$ . It is shown in [2] that this correspondence is 1-1. By our choice of surfaces there are

$$(S_{i_1} \cdot S_{i_2})(S_{i_3} \cdot S_{i_4}) + (S_{i_1} \cdot S_{i_3}) + (S_{i_1} \cdot S_{i_4})(S_{i_2} \cdot S_{i_3}) \equiv 1 \pmod{2}$$

such pairs of points.

Donaldson shows that the end of  $N^2$  corresponding to  $(\Theta_X, \{x, y\})$  is the cone on a circle  $L_{x,y}$  and further that  $[L_{x,y}] \neq 0$  in  $H_1(\mathcal{B}_X^\dagger; \mathbf{Z}_2)$ . This is proved by producing a class  $u_1 \in H^1(\mathcal{B}_X^\dagger; \mathbf{Z}_2)$  which evaluates nontrivially on each  $[L_{x,y}]$ . Since there is an odd number of these links, there must be other ends of  $N^2$ , and each of these other ends must correspond to a sequence of "instantons travelling down the tube"  $\Sigma \times [0, \infty)$  in  $X_+$  together with instantons at some points of  $X$  as above. This means that there is a corresponding sequence  $\{A_n\}$  in  $N^2$  such that for large enough  $n$ ,  $A_n$  is close to a grafted connection  $I_1 \# \dots \# I_k \# A_X \# B_1 \# \dots \# B_l$ , where each  $I_i$  is an instanton at  $x_i \in X$ ,  $A_X \in \mathcal{M}_{X_+}(\rho)$  for some flat connection  $\rho$  on  $\Sigma$ , and there are flat connections  $\rho_j$  on  $\Sigma$ ,  $j = 0, 1, \dots, l$ , such that  $\rho_0 = \rho$ ,  $\rho_l = \theta$ , and  $B_j \in \mathcal{M}_{\pm\Sigma}(\rho_{j-1}, \rho_j)$ , where  $\dim \mathcal{M}_{\pm\Sigma}(\rho_{j-1}, \rho_j) \geq 1$ . (See [5, 1.c.2].)

A dimension count quickly clarifies this situation. The sum of the dimensions of the moduli spaces containing the  $I_i$ ,  $A_X$ , and  $B_j$  must be less than or equal to  $10 = \dim \mathcal{M}_{X_+,2}(\theta)$ . It is clear that  $k \leq 2$ . If  $k = 2$  then we are left as above with limit the trivial connection, and there is no energy left for instantons to travel down the tube giving  $B_1, \dots, B_l$ . If  $k = 1$  then as before  $A_X \in V_{S_{i_1}} \cap V_{S_{i_2}} \cap \mathcal{M}_{X_+,1}(\rho)$ . This is also impossible since by hypothesis  $B_1$  is nontrivial; so  $\dim \mathcal{M}_{X_+,1}(\rho) \leq \mathcal{M}_{X_+,1}(\theta) = 2$ . Hence, there are no point instantons in the limit. Thus,  $A_X$  lies in the transverse intersection  $V_{S_1} \cap \dots \cap V_{S_4} \cap \mathcal{M}_{X_+}(\rho)$ . This means that  $\dim \mathcal{M}_{X_+}(\rho) \geq 8$ . Since  $B_1$  is nontrivial, it follows from Proposition 2.3 that  $\pm\Sigma = \Sigma$ ,  $\rho = \alpha$ ,  $\rho_1 = \theta$ , and  $l = 1$ . Thus this end of  $\mathcal{M}_{X_+,2}(\theta)$  is related to a local diffeomorphism

$$\mathcal{M}_{X_+}(\alpha) \times \mathcal{M}_\Sigma(\alpha, \theta) \rightarrow \mathcal{M}_{X_+,2}(\theta),$$

where  $\dim \mathcal{M}_{X_+}(\alpha) = 8$  and  $\dim \mathcal{M}_\Sigma(\alpha, \theta) = 2$ .



It follows that  $\mathcal{M}_{X_+}(\alpha) \cap V_{S_1} \cap \dots \cap V_{S_4}$  is a 0-dimensional submanifold, and is compact by another codimension argument. Hence,  $\mathcal{M}_{X_+}(\alpha) \cap V_{S_1} \cap \dots \cap V_{S_4}$  is a finite set, say equal to  $\{A_i: i = 1, \dots, r\}$ . Furthermore the existence of a temporal gauge shows that  $\mathcal{M}_\Sigma(\alpha, \theta) = \widehat{\mathcal{M}}_\Sigma(\alpha, \theta) \times \mathbf{R}$ , where  $\widehat{\mathcal{M}}_\Sigma(\alpha, \theta)$  is a 1-manifold. Any sequence of connections  $\{B_n\}$  in  $\widehat{\mathcal{M}}_\Sigma(\alpha, \theta)$  which fails to converge will correspond to a local diffeomorphism

$$\widehat{\mathcal{M}}_\Sigma(\alpha, \sigma_1) \times \mathbf{R} \times \widehat{\mathcal{M}}_\Sigma(\sigma_1, \sigma_2) \times \mathbf{R} \times \dots \times \widehat{\mathcal{M}}_\Sigma(\sigma_k, \theta) \rightarrow \widehat{\mathcal{M}}_\Sigma(\alpha, \theta).$$

Proposition 2.3 together with a simple dimension count implies that this does not occur, and so  $\widehat{\mathcal{M}}_\Sigma(\alpha, \theta)$  is compact. Say that  $\widehat{\mathcal{M}}_\Sigma(\alpha, \theta)$  is the disjoint union of components  $\{S_i^1\}_{i=1}^t$ . The end of  $N^2$  corresponding to  $\{A_i\} \times S_j^1$  is  $A_i \# S_j^1 = \{A_k \# B: B \in S_j^1\}$ , and the characteristic class  $u_1$  evaluates nontrivially on an odd number of these.

Donaldson defines the class  $u_1$  as follows. For each connection  $A \in \mathcal{B}_{X_+}^*$  we can twist the Dirac operator over  $X_+$  to get a family of operators  $\mathcal{D}_A$  over  $\mathcal{B}_{X_+}^*$ . Since the bundles in question (i.e., the bundle supporting  $A$  and the  $\pm$  spin bundles) have structure group  $SU(2) \cong Sp(1)$ , it follows that the index bundle  $\text{Ind } \mathcal{D}_A$  has a real structure. The class  $u_1$  is defined to be  $w_1(\det(\text{Ind } \mathcal{D}_A))$ . (The descent of the real line bundle to  $\mathcal{B}_{X_+}^*$  follows from the fact that the numerical index  $\text{ind } \mathcal{D}_A$  is even; see [2].) To evaluate  $u_1$  on an end  $A_i \# S_j^1$  of  $N^2$ , first restrict the real line bundle  $\det(\text{Ind } \mathcal{D}_A)$  over  $A_i \# S_j^1$ . Since  $A_i$  is fixed, as an element of  $K(A_i \# S_j^1)$ , the index  $\text{Ind } \mathcal{D}_A$  has a constant contribution (the numerical index) coming from  $A_i$  and a perhaps twisted contribution from  $S_j^1$ . (This can be seen by an excision argument exactly as in [2, Lemma 3.24].) Thus there is a real line bundle over  $S_j^1$  whose first Stiefel-Whitney class  $v(S_j^1) \in H^1(S_j^1; \mathbf{Z}_2)$  satisfies  $u_1|_{A_i \# S_j^1} = w_1(\det(\text{Ind } \mathcal{D}_A|_{A_i \# S_j^1})) = v(S_j^1)$  for each  $i = 1, \dots, r$  where  $H^1(A_i \# S_j^1; \mathbf{Z}_2)$  is identified with  $H^1(S_j^1; \mathbf{Z}_2)$ . If  $v(S_j^1) \neq 0$  for  $j = 1, \dots, s$ , and  $v(S_j^1) = 0$  for  $j = s + 1, \dots, t$ , then  $rs$  is the number of ends  $A_i \# S_j^1$  of  $N^2$  on which  $u_1$  evaluates nontrivially. So  $rs$  is odd and therefore  $r$  is odd. Hence we have:

**Proposition 3.1.** *There is a nonempty moduli space  $\mathcal{M}_{X_+}(\alpha)$  of  $SU(2)$  anti-self-dual connections over  $X_+$  such that  $\dim \mathcal{M}_{X_+}(\alpha) = 8$  and  $m_\alpha = \#(V_{S_1} \cap \dots \cap V_{S_4} \cap \mathcal{M}_{X_+}(\alpha)) \equiv 1 \pmod{2}$ .*

As we mentioned earlier, the key idea in the proof of Proposition 3.1 is that Donaldson's proof of his Theorem B [2], which shows that there are no closed simply connected 4-manifolds with intersection form  $pE_8 \oplus H$  for  $p > 0$ , does not contradict the existence of  $E_8 \oplus H$  on  $X_+$  but rather gives information about the ends of  $\mathcal{M}_{X_+,2}(\theta)$  which correspond to the end of  $X_+$ . Similarly, by applying the proof of Donaldson's Theorem C [2], which shows that there are no closed simply connected 4-manifolds with intersection form  $pE_8 \oplus 2H$  for  $p > 0$ , we get information about the ends of  $\mathcal{M}_{Y_-,3}(\theta)$  corresponding to the end of  $Y_-$ . Since Proposition 3.1 has determined that  $\partial X = \Sigma$ , it follows that  $\partial Y = -\Sigma$ . Now using the proof of Donaldson's Theorem C in an argument completely analogous to that of Proposition 3.1 we obtain:

**Proposition 3.2.** *There is a nonempty moduli space  $\mathcal{M}_{Y_-}(\alpha)$  of  $SU(2)$  anti-self-dual connections over  $Y_-$  such that  $\dim \mathcal{M}_{Y_-}(\alpha) = 12$  and  $n_\alpha = \#(V_{S_5} \cap \cdots \cap V_{S_{10}} \cap \mathcal{M}_{Y_-}(\alpha)) \equiv 1 \pmod{2}$ .*

Thus Proposition 2.5 now implies

**Theorem 3.3.** *Suppose  $M$  is a homotopy K3 surface containing  $\Sigma(2, 3, 7)$ . Then its Donaldson polynomial invariant  $q_4(M)(S_1, \dots, S_{10}) \equiv 1 \pmod{2}$ .*

As a corollary we have Theorem 1.1.

It is natural to ask whether  $q_4(M)(S_1, \dots, S_{10}) \equiv 1 \pmod{2}$  in any homotopy K3 surface.

## References

- [1] S. Akbulut, *A fake compact contractible 4-manifold*, J. Differential Geometry **33** (1991).
- [2] S. Donaldson, *Connections, cohomology, and the intersection form of 4-manifolds*, J. Differential Geometry **24** (1986) 275–341.
- [3] ———, *Polynomial invariants for smooth 4-manifolds*, Topology **29** (1990) 257–315.
- [4] R. Fintushel & R. Stern, *Instanton homology of Seifert-fibered homology three spheres*, Proc. London Math. Soc. (3) **61** (1990) 109–137.
- [5] A. Floer, *An instanton invariant for 3-manifolds*, Comm. Math. Phys. **118** (1988) 215–240.
- [6] D. Freed & K. Uhlenbeck, *Instantons and four-manifolds*, Math. Sci. Research Inst. Series, vol. 1, Springer, Berlin, 1984.
- [7] K. Kodaira, *On homotopy K3 surfaces*, Essays on Topology and Related Topics, Mémoires dédiés à Georges deRham, Springer, 1970, 58–69.
- [8] E. Looijenga, *The smoothing components of a triangle singularity*. I, Proc. Sympos. Pure Math., Vol. 40, Amer. Math. Soc., Providence, RI, 1983, 173–184.
- [9] T. Mrówka, *A local Mayer-Vietoris principle for Yang-Mills moduli spaces*, Ph.D. Thesis, Berkeley, 1988.

- [10] C. Taubes, *Self-dual connections on 4-manifolds with indefinite intersection matrix*, J. Differential Geometry **19** (1984) 517–560.
- [11] —, *Gauge theory on asymptotically periodic 4-manifolds*, J. Differential Geometry **25** (1987) 363–430.
- [12] K. Uhlenbeck, *Connections with  $L^p$  bounds on curvature*, Comm. Math. Phys. **83** (1982) 31–42.

MICHIGAN STATE UNIVERSITY  
UNIVERSITY OF CALIFORNIA, IRVINE